Lecture 6. Elasticity

Beyond standard elastic behavior: Metamaterials

The thermodynamics of stress and strain

The first law of thermodynamics (conservation of energy) for a system with a fixed number of particles is:

$$dE = TdS \underbrace{- p\,dV}_{\text{work done on the system}}$$

In the case of a deformation, the amount of work done on a piece of material when deforming it by an amound du_i is equal to the force × displacent, so per unit volume we have:

$$dw = f_i \, du_i$$

We can express the force per unit volume as the divergence of the stress tensor (lecture 3), so the total amount of work done on the material when deforming it is:

$$dW = \int dw\, dV = \int \frac{\partial \sigma_{ij}}{\partial x_j} du_i \, dV$$

To calculate this integral we consider the divergence of the product $\sigma_{ij} du_i$:

$$\frac{\partial}{\partial x_j}(\sigma_{ij} du_i) = \frac{\partial \sigma_{ij}}{\partial x_j} du_i + \sigma_{ij} \frac{\partial du_i}{\partial x_j} \Rightarrow$$

$$\int_V \frac{\partial}{\partial x_j}(\sigma_{ij} du_i)\, dV = \int_{\partial V} \sigma_{ij} du_i n_j \, dA = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} du_i \, dV + \int_V \sigma_{ij} \frac{\partial}{\partial x_j} du_i \, dV$$

$$\Rightarrow \int_V \frac{\partial \sigma_{ij}}{\partial x_j} du_i \, dV = \underbrace{\int_{\partial V} \sigma_{ij} du_i n_j \, dA}_{\to 0} - \int_V \sigma_{ij} \frac{\partial du_i}{\partial x_j} dV = -\frac{1}{2} \int \sigma_{ij} \underbrace{\left(\frac{\partial du_i}{\partial x_j} + \frac{\partial du_j}{\partial x_i} \right)}_{\gamma_{ij}} dV$$

considering an infinite medium or any domain where du_i vanishes at the boundary i.e. the deformation is negligible at large distances.

$$\therefore \; dW = -\frac{1}{2} \int \sigma_{ij} \, d\gamma_{ij} \, dV \quad \text{work done on the volume } dV$$

So the internal energy ~~for a deformed solid becomes~~ per unit volume for a deformed solid becomes:

$$d\mathcal{E} = Tds + \frac{1}{2} \sigma_{ij} d\gamma_{ij}$$ where $\mathcal{E} = \frac{E}{V}$ and $s = \frac{S}{V}$ are the energy and entropy per unit volume.

Rather than considering the free energy of the system, it's useful to know the amount of energy that is actually available to do work, i.e the Helmholtz free energy $F = E - TS$.
~~The differential form is given by~~ of the Helmholtz free energy per unit volume is:

$$df = -s\,dT + \frac{1}{2} \sigma_{ij} \, d\gamma_{ij}$$, which follows directly from the expression for $d\mathcal{E}$

So given the internal energy or the free energy we can calculate the stress of a system by taking the derivative of the energy with respect to the strain:

$$\sigma_{ij} = 2\left(\frac{\partial \varepsilon}{\partial \gamma_{ij}}\right)_S = 2\left(\frac{\partial f}{\partial \gamma_{ij}}\right)_T$$

Free energy of deformations.

For most solid materials the strain is small even when applying large stresses so we can get a reasonable approximation of the free energy by expanding it in terms of strain:

$$f \simeq \underbrace{\cancel{f_0}}_{\text{arbitrary choice}}^{0} + \left(\frac{\partial f}{\partial \gamma_{ij}}\right)_0 \gamma_{ij} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \gamma_{ij}\partial \gamma_{kl}}\right)_0 \gamma_{ij}\gamma_{kl} + \ldots$$

To see the linear term vanishes we consider the stress:

$$\sigma_{ij} = 2\left(\frac{\partial f}{\partial \gamma_{ij}}\right) = 2\left[\left(\frac{\partial f}{\partial \gamma_{ij}}\right)_0 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \gamma_{ij}\partial \gamma_{kl}}\right)\gamma_{kl} + \ldots\right]$$

We want to impose the condition that ~~at~~ zero strain implies zero stress so $\left(\frac{\partial f}{\partial \gamma_{ij}}\right)_0 = 0$

Then the first non-zero term in the free energy expression is the quadratic one so:

$$f \simeq \frac{1}{2}\underbrace{\left(\frac{\partial^2 f}{\partial \gamma_{ij}\partial \gamma_{kl}}\right)}_{C_{ijkl}\ \equiv\ \text{elastic modulus tensor or stiffness tensor}}\gamma_{ij}\gamma_{kl} = \frac{1}{2} C_{ijkl}\gamma_{ij}\gamma_{kl}$$

We can differentiate the free energy to get the relation between stress and strain:

$$\sigma_{ij} = 2\left(\frac{\partial f}{\partial \gamma_{ij}}\right) = C_{ijkl}\gamma_{kl}$$

Hook's law in tensor form (general case)

Now if we consider an isotropic material (i.e. it looks the same in all directions), the stress tensor cannot depend on the material's orientation. Therefore in the isotropic case the free energy can only depend on scalars invariant under rotations

We can construct two such invariant scalars from the strain tensor that are quadratic in $\underline{\underline{\gamma}}$:

$$\underbrace{[\mathrm{Tr}(\underline{\underline{\gamma}})]^2 = \gamma_{ii}^2}_{\text{the square of the trace}} \quad \text{and} \quad \underbrace{\mathrm{Tr}(\underline{\underline{\gamma}}^2) = \gamma_{ij}\gamma_{ij} = \gamma_{ij}^2}_{\text{the sum of the square of all elements}}$$

So the most general expression for f to second order in $\underline{\underline{\gamma}}$ is:

$$f = \frac{1}{2}\lambda\gamma_{ii}^2 + \mu\gamma_{ij}^2$$

where λ, μ are the Lamé coefficients of a solid material

We can relate the Lamé coefficients to more intuitive quantities like the bulk and shear modulus of the material. To do this we consider that any deformation ~~can be~~ is a combination of pure shear and a hydrostatic compression, which means we can write the strain tensor as:

$$\gamma_{ij} = \underbrace{\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)}_{\text{pure shear}} + \underbrace{\frac{1}{3}\gamma_{kk}\delta_{ij}}_{\text{pure compression}}$$

If we substitute this expansion into the expression for the free energy we get:

$$f = \left(\frac{1}{2}\lambda + \frac{1}{3}\mu\right)\gamma_{ii}^2 + \mu\left[\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right]^2$$

In the case of a pure shear the trace of the strain tensor vanishes, and calculating the stress tensor by taking the derivative of the free energy yields:

$$\sigma_{ij} = \frac{2\partial f}{\partial \gamma_{ij}} = \underbrace{4\mu}_{G}\gamma_{ij} = G\gamma_{ij}$$ where G is the shear modulus (by definition).

Likewise for a pure compression, the second term in the free energy expression vanishes and calculating the stress tensor by taking the derivative of the free energy yields:

$$\sigma_{ij} = \frac{2\partial f}{\partial \gamma_{ij}} = 4\left(\frac{1}{2}\lambda + \frac{1}{3}\mu\right)(\delta_{ij}\gamma_{ij})\delta_{ij} = 8\underbrace{\left(\frac{1}{2}\lambda + \frac{1}{3}\mu\right)}_{K}\frac{\Delta V}{V}\delta_{ij} = K\frac{\Delta V}{V}\delta_{ij}$$ where K is the bulk modulus

using $\gamma_{kk} = \delta_{ij}\gamma_{ij}$

the trace of $\underline{\gamma}$ is twice the relative change in volume

by definition of K

So in terms of the shear and bulk moduli the free energy is:

$$F = \frac{1}{8}K\gamma_{kk}^2 + \frac{G}{4}\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)^2 \quad (*)$$

Since the undeformed state must be a minimum of f, we conclude that both K & G > 0.

Note that because the free energy is quadratic in the strain, we can make use of the general property of quadratic functions: if $f(x) = \frac{1}{2}ax^2 \Rightarrow x\frac{\partial f}{\partial x} = ax^2 = 2f(x)$. Applying this to the free energy gives us:

$$2f = \gamma_{ij}\frac{\partial f}{\partial \gamma_{ij}} = \frac{1}{2}\gamma_{ij}\sigma_{ij} \Rightarrow \boxed{f = \frac{1}{4}\gamma_{ij}\sigma_{ij}}$$ ← Elastic energy density associated w/ the stress & strain fields.

For a harmonic spring with force $F = -kx$ the energy associated with it is $U = \frac{1}{2}kx^2$, the elastic energy associated w/ the spring is $\mathcal{E}_{elas} = \frac{1}{2}\sigma_{ij}\gamma_{ij}$

Hooke's law

From the expression of the free energy we arrived to ~~calculating~~ for a homogeneous material with small strains we can obtain a relation between the stress and strain (Hooke's law)

We calculate the differential of (*):

$$\begin{aligned}
df &= \frac{1}{4}K\gamma_{kk}d\gamma_{kk} + \frac{1}{2}G\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)d\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right) \\
&= \frac{1}{4}K\gamma_{kk}\delta_{ij}d\gamma_{ij} + \frac{1}{2}G\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)d\gamma_{ij} \\
&= \frac{1}{2}\sigma_{ij}\,d\gamma_{ij}
\end{aligned}$$

$$\therefore \boxed{\sigma_{ij} = \frac{1}{2}K\gamma_{kk}\delta_{ij} + G\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)}$$ where K ≡ bulk modulus, G ≡ shear modulus.

Hooke's law for isotropic materials

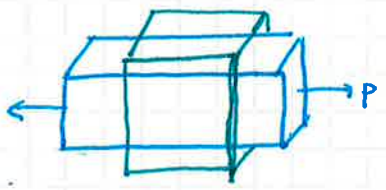
Note $[K] = [G] = \frac{\text{Force}}{\text{Area}} = \frac{kg}{ms^2}$

For future reference it's useful to write the strain in terms of the stresses. To do so we can invert the last equation and obtain:

$$\boxed{\gamma_{ij} = \frac{2}{9}\frac{1}{K}\sigma_{kk}\delta_{ij} + \frac{1}{G}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)}$$

stretching rods

Let's consider a bar of an isotropic elastic medium, on which we pull on both sides in the $\hat{x}$ direction



because the rod is uniform, the resulting deformation is homogeneous, which means γ_{ij} is constant through the rod, so the condition for static equilibrium is:

$$\partial_j \sigma_{ij} = 0$$

$$\sigma_{ij} n_j = P_i$$

For an extension in the $\hat{x}$ direction we have $P_z = P_y = 0$
Since the outward surface normal is perpendicular to the $\hat{x}$ direction on both sides all components of $\underline{\underline{\sigma}}$ will vanish except σ_{xx}, so we have

$$\sigma_{xx} = P \quad , \quad \sigma_{ij} = 0 \quad \text{for al } i,j \neq xx$$

From the inverted Hooke's law we derived we can find immediately the strain:

$$\gamma_{zz} = \gamma_{yy} = -\frac{1}{3}\left(\frac{1}{G} - \frac{2}{3K}\right)P \quad (*)$$

$$\gamma_{xx} = \frac{2}{3}\left(\frac{1}{G} + \frac{1}{3K}\right)P$$

$$\gamma_{ij} = 0 \quad \text{for } i \neq j$$

The coefficient relating the extension γ_{xx} to the pressure is the Young's modulus E_Y

$$\gamma_{xx} = 2\frac{P}{E_Y} \quad \text{so} \quad E_Y = \frac{9GK}{G+3K}$$

from this we can see that the larger the Young's modulus, the stiffer the material

Now if we consider the contraction in the transverse direction that results from the extensional force in the $\hat{x}$ direction:

$$\gamma_{zz} = \gamma_{yy} = -\nu \gamma_{xx}$$ where ν is the poisson ratio.

Poisson ratio $$\nu \equiv \frac{-\gamma_{yy}}{\gamma_{xx}} = \frac{1}{2}\left(\frac{3K - 2G}{3K + G}\right)$$

measure of the change in the transverse direction relative to the change in the main stretching or compression direction.

Both elastic constants (G & K) are positive so as the ratio $\frac{K}{G}$ is varied the Poisson ratio can vary between -1 and 0.5, with $\nu \sim 0.5$ in the limit $K \gg G$ and $\nu \simeq -1$ for $K \ll G$

The expressions for K and G in terms of the Poisson ration and Young's modulus can be obtained straightforwardly from the above equations:

$$K = \frac{E_Y}{3(1-2\nu)} \quad , \quad G = \frac{E_Y}{2(1+\nu)}$$

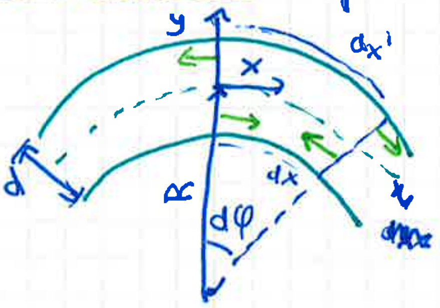
Most materials with $\nu > 0$ contract (stretch) in the transverse directions if stretched (squeezed) in one direction. Typical materials have $\nu \simeq 0.33 \Rightarrow$ they contract about 1/3 of the amount by which they are stretched in the transverse direction. One exception is cork, which has $\nu \simeq 0$
Why does it make it a good material for wine bottle stoppers?

Bending rods

For many everyday objects e.g. a ruler or a sheet of paper, a stretching deformation requires a large amount of force. However it's much easier to bend these objects.

We will now consider a bending deformation. These deformations are part stretching and part compressing. We will look at what the bending implies for the elastic distortions within a rod.

For an isotropic and homogeneous medium there is a neutral surface halfway through the rod which is neither stretched or compressed



$K < 0$

we assume only one radius of curvature, the object is bent in only one direction

- outer half of the rod is stretched
- inner half of the rod is compressed
- dotted line is the neutral surface
- → ← forces on a surface element of the cross-section
- These forces will also give rise to a torque
- $\frac{d}{R} \ll 1$ local radius of curvature is much larger than object thickness

We'll consider a local coord system where x is parallel to the neutral surface in the direction of bending, and y in the direction perpendicular to it, with y increasing outward

The relative stretching of a little material element in the x direction along the rod is proportional to the increase in length of each circular element:

$$\gamma_x \simeq \frac{dx' - dx}{dx} = \frac{(R+y)\,d\varphi - R\,d\varphi}{R\,d\varphi} = \frac{y}{R}$$

we can see that for $y > 0$ we have a stretching & for $y < 0$ we have a compression

since this strain represents a compression or stretching (depending on the sign of x), the corresponding stress is simply the Young's modulus times the strain:

$$\sigma = \frac{E_Y\, y}{R}$$

so we have that:

$$\boxed{\begin{array}{ll} \gamma_x = \dfrac{y}{R} & \text{bending strain} \\[2ex] \sigma_x = \dfrac{E_Y\, y}{R} & \text{bending stress} \end{array}}$$

As we discussed before, if a medium is stretched or compressed in one direction, so does the transverse direction, depending on the Poisson ratio.

The material is free to expand or contract in the y direction since the surfaces of the rod are free $\Rightarrow \sigma_{yy} \simeq 0$

We could get the force acting on the piece of rod by integrating the expression for σ_x over the cross sectional area of the rod.

However, we're more interested in the torque exerted by this force ~~per~~. Because we are bending around the z axis, and the stretching/compression forces point along the $\hat{x}$ axis, the ~~[illegible]~~ lever arm points along the y direction and ~~[illegible]~~ so the torque per unit area is:

$$-\sigma y = -E_Y \frac{y^2}{R}$$

The total torque known as the bending moment of the beam can be obtained by integrating over the cross sectional area of the rod:

$$M_z = -\frac{E_Y}{R}\underbrace{\int y^2 dA}_{I_z \equiv \text{second moment of area}} = \frac{-E_Y I_z}{R}$$

we can rewrite this equation in terms of a function describing the shape of the beam $w(x)$

$$M_z(x) = -E_Y I_z \underbrace{\frac{d^2 w}{dx^2}}_{K \equiv \text{curvature}} \quad (*)$$

similar to the moment of inertia → consists of an (area) a distance to a reference axis squared.

Bending energy of a rod

We will now analyze the elastic bending energy E_{bend} associated with the bending.
We derived earlier that the elastic energy density is:

$$E = \frac{1}{4}\sigma_{ij}\gamma_{ij}$$

Using this expression we can calculate the elastic bending energy for a small area element of the rod parallel to the neutral surface

$$\frac{E_{bend}}{area} \simeq \frac{1}{4}\int_{-d/2}^{d/2}\sigma_{ij}\gamma_{ij}\, dy \simeq \frac{1}{4}E_Y f_b(\nu)\int_{-d/2}^{d/2}\left(\frac{y}{R}\right)^2 dy = \frac{E_Y d^3 K^2}{24(1-\nu^2)}$$

where $K = \frac{1}{R}$

$f_b(\nu)$ function of the Poisson ratio

We have analyzed the contribution to the elastic energy with local curvature $K = \frac{1}{R}$.

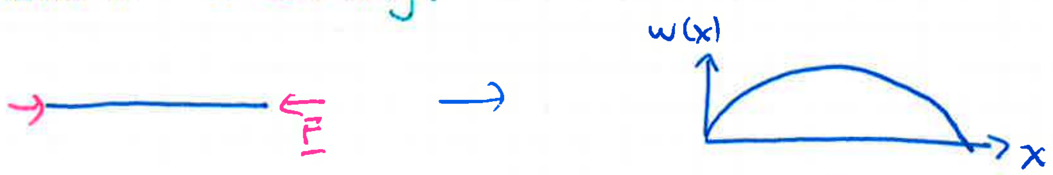
For a deformed rod K will vary along the rod, so based on the above analysis the total elastic bending energy of the deformed bar is:

$$E_{bend} \simeq \frac{E_Y d^3}{24(1-\nu^2)}\int K^2 ds$$

where the $\int$ is taken along the neutral surface.

Buckling of a thin rod

When a long rod is compressed by exerting an external force it starts to bend, this phenomenon is called buckling.



Before buckling the bending moment of the rod is zero. After buckling the bending moment is given by $M_z = Fw(x)$. By substituting this in equation (*) we get:

$$Fw(x) = -E_Y I_z \frac{d^2 w}{dx^2} \Rightarrow \frac{d^2 w(x)}{dx^2} = -\frac{F}{EI_z}w(x)$$

The general solution of this eq is a linear combination of a sine + cosine. Defining $\lambda = \sqrt{F/EI_z}$ we get:

$$w(x) = A\sin(\lambda x) + B\cos(\lambda x)$$

L = length of the rod.

If we fix the end points of the beam we have that $w(0) = w(L) = 0$. From the first condition we get $B = 0$, from the second $A = 0$ (beam hasn't buckled) or $\lambda L = n\pi$, with n an integer

For the simplest case: we get $w(x) = w_{max} \sin\left(\frac{\pi x}{L}\right)$, by subs this into the equation for w we can get an expression relating the buckling force to the rod's length:

$$F_{buckle} = \frac{\pi^2 E I_z}{L^2}$$

The exact shape of the buckled rod depends on the boundary conditions. The scaling of the buckling force is the same for other boundary conditions, only a small numeric factor changes. We can then introduce K, known as column effective length factor, with the inclusion of K:

$$F_{buckle} = \frac{\pi^2 E I_y}{(KL)^2}$$

with $K = 1$ two hinged ends (free to rotate)
$= 0.5$ two fixed ends
$= 2$ one fixed end and one free to move in any lateral direction